Math 102

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Announcements

- Midterm solutions posted
- Your midterm viewable in Crowdmark.
  - Remember, the midterm is an excellent way to practice in preparation for the final exam.
  - OSH will require you to reflect on your midterm, I highly urge you to take time to do this.
- Midterm statistics. Both out of 50.
  - Version 1: Mean 26.95, Median 28, StdDev 9.312
  - Version 2: Mean 28.13, Median 29, StdDev 8.765

Goals Today

- What is a differential equation?
  - What's a solution?
  - $\frac{df}{dt} = kf$
  - Solving with initial conditions.
- Qualitative Analysis for Autonomous DE's
  - Steady States
  - Qualitative analysis
- Writing down DE's

Differential Equations

Algebraic equation	Differential Equation
$2x + 5 = 0$	$\frac{df}{dt} = 4f(t) - 3$
$x^3 + 3x = 2$	$\frac{df}{dt} - (f(t))^2 + 3f(t) = 0$

- A differential equation is an equation involving the derivative of a function.
- A solution to a differential equation is any function $f(t)$ which makes the equation true.
- Most typically, differential equations relate $f(t)$ to its derivative. Generally it's hard to **find** a solution. But it's easy to **check** whether a given function is a solution or not.

Exercise: Consider the differential equation

$$\frac{df}{dt} = 2t$$

Prove that $f(t) = t^2 - 2$ is a solution.

Exercise: Consider the differential equation

$$\frac{df}{dt} = 3f$$

Prove that $f(t) = 4e^{3t}$ is a solution.

Here we write f to mean $f(t)$: we have dropped the '(t)' from the notation for brevity.

The differential equation $\frac{df}{dt} = kf$

Let k be a constant. Then the differential equation

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has $f(t) = e^{kt}$ as a solution.

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Question: Can you come up with other solutions?
How about a way to describe ALL of the solutions?

The general solution to a DE

$$\frac{df}{dt} = 2t$$

The general solution to this differential equation is $f(t) = t^2 + C$ where C is an arbitrary constant.

$$\frac{df}{dt} = kf$$

The general solution to this differential equation is $f(t) = Ce^{kt}$ where C is an arbitrary constant.

Initial conditions

The general solution to the differential equation

$$\frac{df}{dt} = 3f$$

is $f(t) = Ce^{3t}$ where C may be any constant.

Question: Find the solution to this differential equation satisfying the initial condition

$$f(0) = \frac{1}{2}$$

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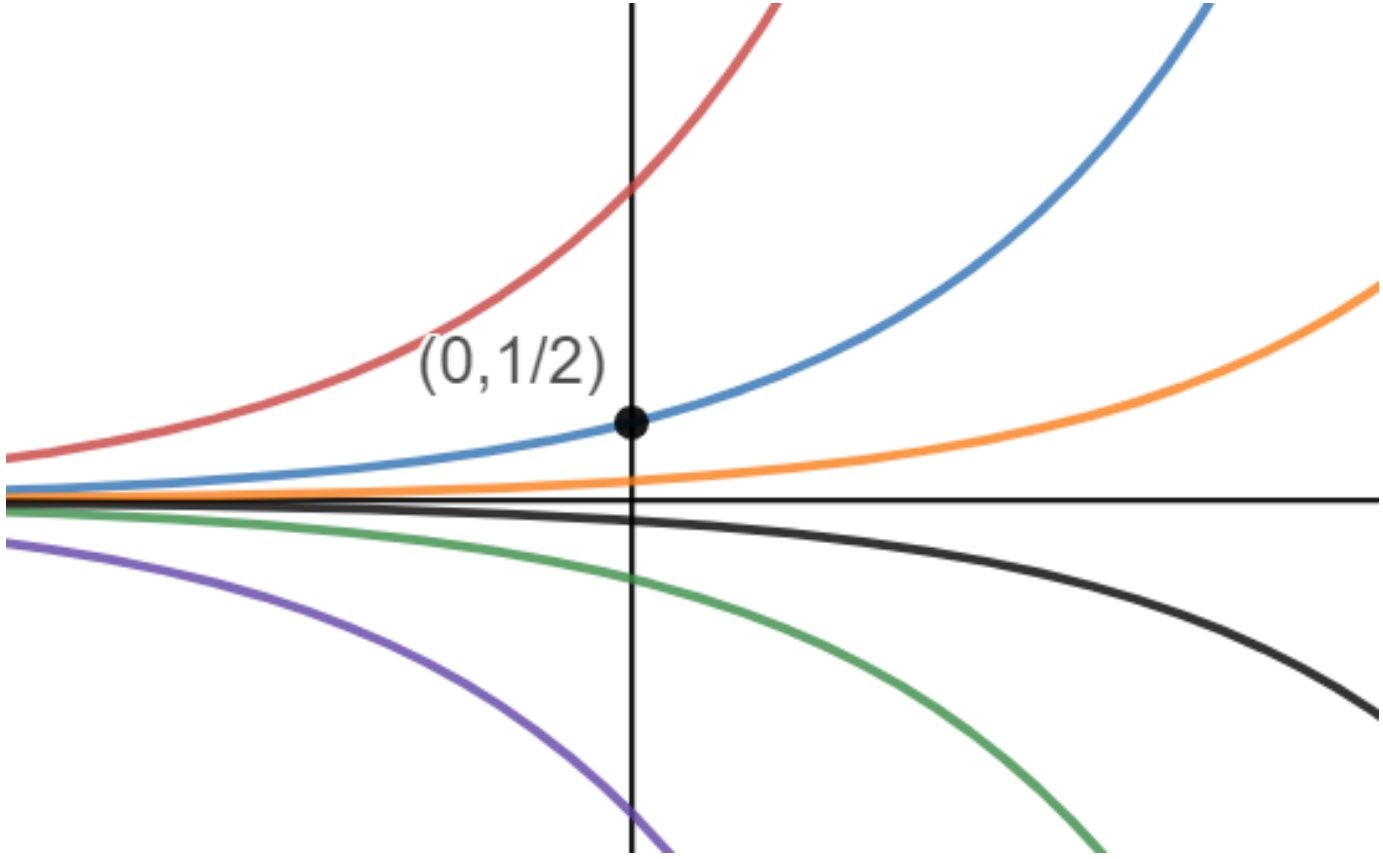
Solution: $f(0) = C \cdot e^{3 \cdot 0} = C \cdot 1 = C$. Thus, $C = \frac{1}{2}$, and it follows that $\boxed{f(t) = \frac{1}{2}e^{3t}}$.



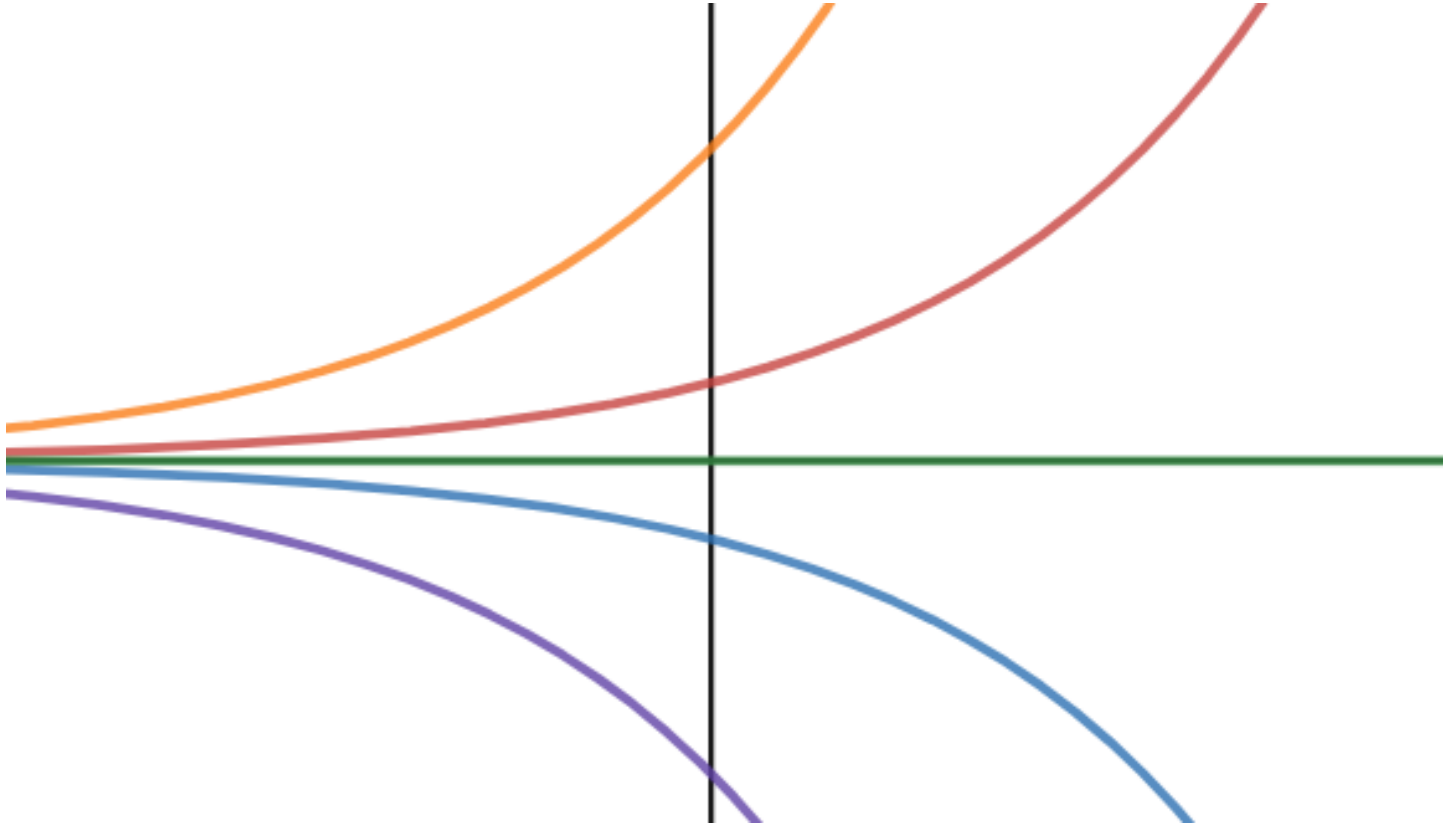
Among all of the solutions, there's only one that passes through $(0, \frac{1}{2})$, namely $f(t) = \frac{1}{2}e^{3t}$.

Initial conditions - practice

Question: Find the solution to $\frac{df}{dt} = -4f$ satisfying $f(\ln(2)) = \frac{1}{6}$.

Question: Find the solution to $\frac{df}{dt} = -4f$ satisfying $f(3) = 0$.

Steady states



A steady state solution is a solution of the form $f(t) = \text{constant}$. You can find the steady states by setting $\frac{df}{dt} = 0$ and solving.

Qualitative Analysis

We will make a sketch of the solutions to the differential equation

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Question: Find all steady states for the differential equation above.

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Solution: If $f(t) = C$ is a steady state, then $\frac{df}{dt} = 0$. Thus, $2C - 6 = 0$, and so $C = 3$. Therefore, $\boxed{f(t) = 3}$ is the only steady state solution.

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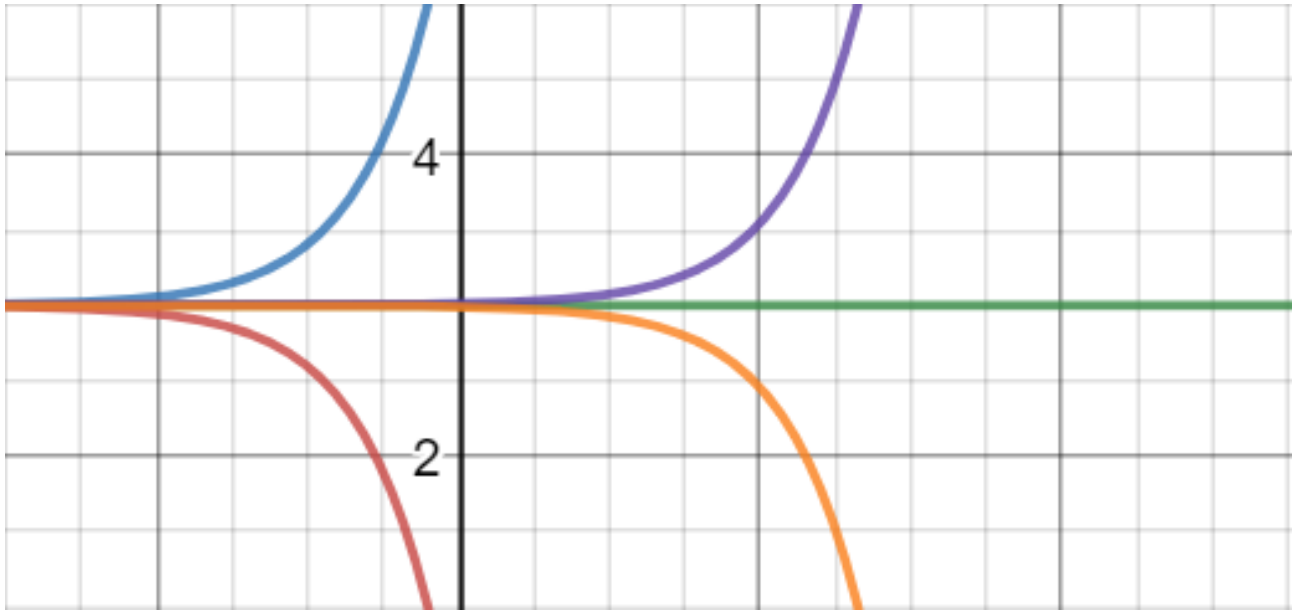

Qualitative Analysis

Question: Consider the differential equation

$$\frac{df}{dt} = -3f + 12$$

Calculate the steady states, and then perform qualitative analysis to sketch the solutions.

Writing a Differential Equation

Question: A drug solution is administered by IV to a patient at a constant rate of 50 ml/hr. At the same time, the drug is metabolized in the body at a rate equal to 0.5 times the volume in the body, per hour.

Let $D = D(t)$ denote the amount of drug in the body at time t. Write down a differential equation for D. Then perform a qualitative analysis on your differential equation.

Recap

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